ALMOST-PERIODIC SOLUTIONS OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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We consider the nonlinear system of differential equations

$$(S_t)\ \frac{dx_s}{dt} = \omega_s(t,\ x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{1}$$

where the functions $\omega_s^p(t,\ x_1,\ \ldots,\ x_n)$ defined in the region $R(-\infty < t < +\infty)$, $|x_s| < M < \infty)$ are continuous in t and satisfy the Lipschitz condition relative to the variables $x_1,\ \ldots,\ x_n$. Under the specified conditions the system (1) has a unique solution satisfying the initial conditions $x_1(0),\ \ldots,\ x_n(0)$.

Let us now consider the sequence of systems of differential equations

$$(S_t^{(p)})\ \frac{dx_s}{dt} = \omega_t^{(p)}(t,\ x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n;\ p = 1, 2, \ldots) \tag{2}$$

where the functions $\omega_s^{(p)}(t,\ x_1,\ \ldots,\ x_n)$ satisfy the same conditions as the function ω_s. Let us assume that in every bounded interval of t the functions $\omega_s^{(p)}$ tend uniformly (in t) to the function ω_s as p goes to infinity. Suppose, furthermore, that the initial values $x_s^{(p)}(0)$ of the solution (2) tend to the limit $x_s(0)$. Then the solutions of the systems $(S_t^{(p)})$ will tend uniformly in every founded interval to the solution of the system (S_t).

We shall assume, further, that the functions $\omega(t,\ x_1,\ \ldots,\ x_n)$ are almost periodic (in the sense of Bohr) in t, uniform with respect to $(x_1,\ \ldots,\ x_n)$ R, i.e. for every given > 0 one can find functions $F_k^{(s)}(x_1,\ \ldots,\ x_n)$ and a real constant $\lambda_k^{(s)}$ such that

$$\left|\omega_s(t,\ x_1, \ldots, x_n) - \sum_k e^{i\lambda_k^{(s)}t} F_k^{(s)}(x_1, \ldots, x_n)\right| < \varepsilon$$

It is known [1] that if the function $\phi(t)$ is almost periodic in the sense of Bohr, then the family of functions $\{\phi(t+h)\}$ is compact in the sense of uniform convergence on the entire real axis. It is not difficult to show that under the assumptions made on the functions $\omega_s(t, x_1, \dots, x_n)$, the family of functions $\{\omega_s(t+h, x_1, \dots, x_n)\}$ is compact in the sense of uniform convergence on the entire axis. Let us consider all possible sequences of real numbers $\{h_k\}$ for which there exist the following limits which are uniform in t:

$$\lim_{k\to\infty} \omega_s(t+h_k, x_1, \dots, x_n) = \omega_s^*(t, x_1, \dots, x_n) \qquad (s = 1, \dots, n)$$

Along with the system (1), we shall consider the systems

$$(S_t^*). \frac{dx_s}{dt} = \omega_s^*(t, x_1, \dots, x_n) \qquad (s = 1, \dots, n)$$

We shall say that the systems (S_{t+h_k}) converge to the system (S_t^*), and we write

$$(S_t^*) = \lim (S_{t+h_k})$$

Selecting different sequences $\{h_k\}$, we obtain the set of systems which we denote by $H(S_{t+h})$. We note that if

$$(S_t^*) = \lim (S_{t+h_k}), \quad \text{то} \quad (S_t) = \lim (S_{t-h_k}).$$

Hence, every one of the systems of the set $H(S_{t+h})$ is determined by some system (S_t^*).

Let us agree to say that two systems (S_t^*) and (S_t^{**}) of the set $H(S_{t+h})$ differ from each other by less than ϵ $(\epsilon > 0)$, if

$$|\omega_s^* - \omega_s^{**}| < \varepsilon \quad (s = 1, 2, \dots, n)$$

In this case we write

$$|(S_t^*) - (S_t^{**})| < \varepsilon$$

Theorem 1. If some system of the set $H(S_{t+h})$ has an almost-periodic solution, then the same thing is true for every system of the set.

Indeed, suppose the system (S_t) has an almost-periodic solution $x_1(t), \dots, x_n(t)$ and suppose (S_t^*) is some other system of the set

$$H(S_{t+h}), \text{ т. е. } (S_t^*) = \lim (S_{t+h_k})$$

For a set of various systems (S_{t+h_k}) $(k = 1, 2, \dots)$ we obtain sets of solutions $x_s(t+h_k)$ $(s = 1, 2, \dots, n;\ k = 1, 2, \dots)$. For a fixed h_k, each solution $x_s(t+h_k)$ $(s = 1, 2, \dots, n)$ is a set of almost-periodic functions. In this manner we obtain n sequences of almost-periodic functions

$$x_s(t+h_1), \qquad x_s(t+h_2), \ldots, \qquad x_s(t+h_k), \ldots$$

But since the almost-periodic functions $x_s(t)$ are normal [1], it follows that one can select from the given sequences uniformly convergent sequences $x_s(t+l_1)$, $x_s(t+l_2)$, ..., $x_s(t+l_k)$, ... $(s = 1, \ldots, n)$, i.e.

$$x_s(t+l_k) \to x_s^*(t) \qquad (-\infty < t < \infty)$$

Since $x_s^*(t)$ is a solution of the system (S_t^*) the theorem is proved.

Note. If $x_s^{(1)}$ and $x_s^{(2)}$ are two distinct almost-periodic solutions of the system (S_t), then one can determine in the same way two almost-periodic solutions of the system (S_t^*) which will also be distinct. From this it follows also that if some system of the set $H(S_{t+h})$ has a unique almost-periodic solution then the same thing is true for each system of the set.

Theorem 2. If each system of the set $H(S_{t+h})$ has a unique bounded solution then this solution consists of almost-periodic functions.

Indeed, the bounded solution $x_s = u_s(t)$ of the system (S_t) is determined by its initial conditions $u_s(0)$. Suppose $(S_t^*) = \lim (S_{t+h_k})$. One may always assume that the numbers $u_s(h_k)$ converge to the limits $u_s^* = \lim u_s(h_k)$. Then the solution $u_s^*(t)$ (of the system (S_t^*)) which for $t = 0$ takes on the system of values u_s^* is bounded and the convergence of $u_s(t+h_k)$ to $u_s^*(t)$ will be uniform on every interval of finite length. In order to prove the theorem one must show that this convergence is uniform on $(-\infty < t < +\infty)$, i.e. that the bounded solution $u_s(t)$ consists of normal functions.

Let us assume that the convergence is not uniform on $-\infty < t < +\infty$. Then we can make the following definitions:

(a) α is a positive number;

(b) $t_1, t_2, \ldots, t_p, \ldots$ is a sequence of numbers which increase in absolute value;

(c) $k_1, k_2, \ldots, k_p, \ldots$, and $r_1, r_2, \ldots, r_p, \ldots$ are two sequences of indices;

(d) the function $u_j(t)$ is one function out of the n functions $u_1(t), \ldots, u_n(t)$ such that

$$|u_j(t_p+h_{k_p}) - u_j(t_p+h_{r_p})| \geqslant \alpha \qquad (p = 1, 2, \ldots) \tag{3}$$

Out of each of the sequences of numbers $u_s(t_p + h_{k_p})$ and $u_s(t + h_{r_p})$, and from each sequence of the system

$$(S_{t+t_p+h_{k_p}}) \text{ и } (S_{t+t_p+h_{r_p}})$$

one can select sequences which have the properties that they converge to the limits $u_s^{(1)}$ and $u_s^{(2)}$, and to $(S^{(1)})$ and $(S^{(2)})$, respectively.

In order not to complicate the notation, let us assume that the above sequences themselves have these properties. The sets of numbers $u_s^{(1)}$ and $u_s^{(2)}$ represent then the initial values of the bounded solutions of the systems $(S^{(1)})$ and $(S^{(2)})$. But it is easy to show that the systems $(S^{(1)})$ and $(S^{(2)})$ coincide. Under our conditions the system $(S^{(1)})$ has only one bounded solution; but we have found two different systems of initial conditions for the system $(S^{(1)})$ and its unique bounded solution. Because of (3) we must have $|u_j^{(1)} - u_j^{(2)}| > a$. This contradiction proves the theorem.

Consequence. Let us consider the autonomous system

$$\frac{dx_s}{dt} = \omega_s(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{4}$$

In this case the set $H(S_{t+h})$ consists of one element. Hence, the following theorem has been established.

Theorem 3. If the system (4) has a unique bounded solution, then this solution consists of almost-periodic (in particular periodic) functions.

Note. Theorems 1 and 2 are extensions to nonlinear systems of theorems that were given by Favard [2] for linear systems of differential equations with almost-periodic coefficients.

In Favard's theorems certain definite conditions have to be satisfied by the entire set of systems of differential equations. Levitan introduced a new class of almost-periodic functions [1], the so-called N-almost-periodic functions. He showed that if one seeks N-almost-periodic solutions of linear systems of differential equations, then one does not have to require the fulfilment of certain definite conditions by the entire set of the systems. A similar situation prevails also for nonlinear systems. It is not difficult to verify, by repeating the proof of Levitan, that the following theorem is valid.

Theorem 4. If the system (S_t) has a unique bounded solution, then this solution consists of N-almost-periodic functions.

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